# PROBLEM SETS FOR INTRODUCTION TO ENUMERATIVE GEOMETRY

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Abstract. This document collects a few problems which should be useful to practice on the material that is covered during the lectures.

## Lecture 1

**Problem 1.** Let $V$ be a vector space of dimension $n+1$ and let $\nu_{d,n} : \mathbb{P}V \to \mathbb{P}S^d V$ be the Veronese embedding. Show that if $X \subseteq \mathbb{P}V$ is a variety of dimension $k$ and degree $e$, then $\nu_{d,n}(X)$ has degree $d^k e$.

In particular, the degree of a $k$-dimensional subvariety of $\nu_{d,n}(\mathbb{P}V)$ is a multiple of $d^k$.

**Problem 2.** Let

$$\Psi = \{(p,q,r) \in \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n : p,q,r \text{ are collinear}\}.$$

Show that $\Psi$ is a subvariety of $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$ of codimension $n-1$.

Determine the class $[\Psi]$ in the Chow ring $\mathrm{CH}(\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n)$.

*Note*: The Chow ring of a product of several projective spaces is what one expects. If $\dim V_j = n_j + 1$, then

$$\mathrm{CH}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_s) = \mathbb{Z}[\alpha_1, \ldots, \alpha_s]/(\alpha_1^{n_1+1}, \ldots, \alpha_s^{n_s+1})$$

where $\alpha_j$ is identified with the class of the pull back of the hyperplane section in $\mathbb{P}V_j$ via the projection map. In other words

$$\alpha_j = [\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_{j-1} \times H_j \times \mathbb{P}V_{j+1} \times \cdots \times \mathbb{P}V_s].$$

**Problem 3.** Let $\mathbb{P}S^3\mathbb{C}^3$ be the space of homogeneous polynomials of degree 3 in three variables. Let

$$\mathcal{T} = \{f : f = \ell_1\ell_2\ell_3 \text{ for some linear forms } \ell_j\},$$

that is the space of *triangles* (i.e., cubic curves which are union of three lines).

Determine the dimension and the degree of $\mathcal{T}$.

*Hint:* Write $\mathcal{T}$ as the image of a map defined on $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$.

**Problem 4.** Let $\mathbb{P}S^3\mathbb{C}^3$ be the space of homogeneous polynomials of degree 3 in three variables, that is the space of plane cubic curves. Let

$$\mathcal{A} = \{f : f = \ell_1\ell_2\ell_3 \text{ for some linear forms } \ell_j \text{ with a common zero}\},$$

that is the space of *asterisks* (i.e., cubic curves which are union of three lines passing through the same point).

Determine the dimension and the degree of $\mathcal{A}$.

*Hint:* It is similar to the previous problem, but one needs something more complicated than $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$.

**Problem 5.** Let $\mathbb{P}S^3\mathbb{C}^3$ be the space of homogeneous polynomials of degree 3 in three variables, that is the space of plane cubic curves. Let

$$\mathcal{C} = \{f = \ell_1^2 \ell_2 : \ell_j \text{ is a linear form}\},$$

that is the space of cubic curves which are union of a double line and a line.

Determine the dimension and the degree of $\mathcal{C}$.

## Lecture 2

**Problem 6.** Let $C_1, C_2 \subseteq \mathbb{P}^3$ be two curves of degree $d_1, d_2$ respectively and genera $g_1, g_2$ respectively. Suppose $C_1, C_2$ are in general position with respect to each other. How many lines are secant both two $C_1$ and $C_2$?

**Problem 7.** Let $C$ be a smooth non-degenerate curve in $\mathbb{P}^3$ of degree $d$ and genus $g$. Let

$$TC = \{\Lambda \in G(2,4) : \Lambda = T_pC \text{ for some } p \in C\}.$$

What is the class of $TC$ in $G(2,4)$?

**Problem 8.** Let $S_1, \ldots, S_4 \subseteq \mathbb{P}^3$ be four surfaces with $\deg(S_i) = d_i$ in general position. How many lines are tangent to all of them?

**Problem 9.** Let $C$ be a smooth non-degenerate curve of degree $d$ and genus $g$ in $\mathbb{P}^3$. Let $S$ be a smooth surface of degree $e$. Suppose $C$ and $S$ are in general position with respect to each other. How many lines are tangent to both $S$ and $C$?

**Problem 10.** Let $X \subseteq G(2,4)$ be an irreducible variety of codimension 2. Then $[X] = \alpha\sigma_2 + \beta\sigma_{1,1}$. Show that if $\alpha = 0$ then $\beta = 1$. What if $\beta = 0$?

## Lecture 3

**Problem 11.** Let $C$ be a smooth non-degenerate curve of degree $d$ and genus $g$ in $\mathbb{P}V$. Let

$$s_2(C) = \overline{\{\Lambda \in G(2,V) : \mathbb{P}\Lambda \text{ is a secant line to } C\}}.$$

Determine the class of $s_2(X)$ in $G(2,V)$.

**Problem 12.** Let $C$ be a smooth non-degenerate curve of degree $d$ and genus $g$ in $\mathbb{P}V$. Let

$$T(C) = \overline{\{\Lambda \in G(2,V) : \mathbb{P}\Lambda \text{ is a tangent line to } C\}}.$$

Determine the class of $T(C)$ in $G(2,V)$.

**Problem 13.** In $G(3,6)$, compute the product

$$\sigma_{2,1} \cdot \sigma_{2,1}.$$

**Problem 14.** Let $\lambda$ be a partition contained in the $k \times (n-k)$ box and let $\Sigma_\lambda$ be the corresponding Schubert variety in $G(k,n)$. Consider the identification

$$i : G(k,n) \to G(n-k,n)$$

Show that $i$ maps $\Sigma_\lambda$ to $\Sigma_{\lambda^T}$, where $\lambda^T$ is the partition in the $(n-k) \times k$ whose Young diagram is the transpose of the Young diagram of $\lambda$.

**Problem 15.** Let $X$ be an irreducible smooth variety of codimension $c$ and degree $d$ in $\mathbb{P}V$. Let

$$H(X) = \{\Lambda \in G(c, V) : \mathbb{P}\Lambda \cap X \neq \emptyset\}.$$

This is the Chow form of $X$.

- Prove that $\operatorname{codim}_{G(c,V)}(H(X)) = 1$;
- Determine the class $[H(X)]$ in $CH^1(G(c, V))$.

## Lecture 4

**Problem 16.** Prove the statements about global sections mentioned during the lectures. In particular prove:

- $H^0(\mathcal{S}) = 0$ where $\mathcal{S}$ is the tautological bundle on $G(k, V)$;
- $H^0(\mathcal{S}^\vee) = V^*$ where $\mathcal{S}^\vee$ is the dual of the tautological bundle on $G(k, V)$;
- $H^0(\mathcal{Q}) = V$ where $\mathcal{Q}$ is the universal quotient bundle on $G(k, V)$.

**Problem 17.** Compute the Chern classes of the tangent bundle of $G(2, 4)$.

**Problem 18.** Let $X$ be a generic hypersurface of degree $2n-3$ in $\mathbb{P}^n$. Prove that $X$ contains a finite number of lines and determine this number.

**Problem 19.** Let $X$ be a generic hypersurface of degree 4 in $\mathbb{P}^7$. Prove that $X$ contains a finite number of 2-planes and determine this number.

**Problem 20.** Let $X_1, X_2$ be two generic cubic hypersurfaces in general position in $\mathbb{P}^5$. How many lines are contained in both of them?

## Lecture 5

**Problem 21.** Let $\mathrm{Sym}_n$ and $\mathrm{Skew}_n$ be the spaces of symmetric and skew-symmetric matrices respectively. Let $A_r = \{S \in \mathrm{Sym}_n : \mathrm{rank}(S) \leq r\}$ and $K_r = \{\Lambda \in \mathrm{Skew}_n : \mathrm{rank}(\Lambda) \leq r\}$. Compute the dimensions of $A_r$ and $K_r$.

**Problem 22.** Let $X_r = \{A \in \mathbb{P}\mathrm{Mat}_{e\times f} : \mathrm{rank}(A) \leq r\}$ be the $r$-th general determinantal variety. Let $M \in X_r$ be a matrix of rank $s \leq r$. Compute

- $T_M X_r$, the tangent space to $X_r$ at $M$;
- $TC_M X_r$, the tangent cone to $X_r$ at $M$.

Deduce that $M$ is a smooth point of $X_r$ if and only if $s = r$.

**Problem 23.** Let $C_d$ be the rational normal curve of degree $d$ in $\mathbb{P}^d$. Let $r < d/2$ and let $\sigma_r(C_d)$ be the $r$-th secant variety of $C_d$. Prove that $\dim \sigma_r(C_d) = 2r - 1$. Is this true for every curve?

**Problem 24.** Let $C_d$ be the rational normal curve of degree $d$ in $\mathbb{P}^d = \mathbb{P}S^d\mathbb{C}^2$. Let $e < d/2$ and let $\mathrm{cat}_e : S^e\mathbb{C}^{2*} \to S^{d-e}\mathbb{C}^2$ be the $e$-th catalecticant map. Show that the $2 \times 2$ minors of $\mathrm{cat}_e$ define $C_d$ set-thereotically (in fact, they generate the ideal). In particular, $C_d$ has the expected dimension as a determinantal variety with respect to $\mathrm{cat}_1$ but it does not have the expected dimension as a determinantal variety with respect to $\mathrm{cat}_e$ for $e \geq 2$.

**Problem 25.** Let $\nu_{2,2}(\mathbb{P}^2)$ be the Veronese surface in $\mathbb{P}^5 = \mathbb{P}S^2\mathbb{C}^3$. Prove that $\sigma_2(\nu_{2,2}(\mathbb{P}^2))$ does not have the expected dimension (as a secant variety).

## Lecture 6

**Problem 26.** Let $C \subseteq \mathbb{P}^3$ be a curve whose ideal is generated by the $2 \times 2$ minors of a $2 \times 3$ matrix

$$B = \begin{pmatrix} g_{0,0} & g_{0,1} & g_{0,2} \\ g_{1,0} & g_{1,1} & g_{1,2} \end{pmatrix}$$

with $g_{ij} \in \mathbb{C}[z_0, z_1, z_2]$ and $\deg(g_{0i}) + \deg(g_{1j}) = \deg(g_{0j}) + \deg(g_{1i})$ (so that the minors are homogeneous). Compute the degree of $C$.

*Hint:* First work out the case where $\deg(g_{0i}) = \deg(g_{1i})$.

**Problem 27.** Solve Problem 2 using Porteous's formula.

**Problem 28.** Let $V$ be a vector space of dimension $n$. Let $k_1, k_2 \leq n$ and consider

$$X_{(k_1,k_2),q} = \{(L, M) \in G(k_1, V) \times G(k_2, V) : L \cap M \geq q\}.$$

Compute $\dim X_{(k_1,k_2),q}$. Realize $X_{(k_1,k_2),q}$ as a degeneracy locus (in a range where it has the expected codimension) and determine its class using Porteous's formula.

**Problem 29.** Let $f_0, f_1 \in S^2\mathbb{C}^3$ be two generic conics. Let $f_\varepsilon = (1-\varepsilon)f_0 + \varepsilon f_1$ be the pencil of conics they generate. Use Porteous's formula to prove that there are two values of $\varepsilon$ for which $f_\varepsilon$ is reducible.

**Problem 30.** Let $f_0, f_1 \in S^4\mathbb{C}^4$ be two generic quartic polynomials. Let $f_\varepsilon = (1-\varepsilon)f_0 + \varepsilon f_1$ be a pencil of quartics, and let $X_\varepsilon = \{f_\varepsilon = 0\}$ be the corresponding quartic surface in $\mathbb{P}^3$. Determine for how many values of $\varepsilon$ the surface $X_\varepsilon$ contains a line.

*Hint:* Represent the pencil as a map of vector bundles $\varphi : \mathcal{O}^{\oplus 2} \to \operatorname{Sym}^4 \mathcal{S}^*$ where $\mathcal{S}$ is the tautological bundle over $G(2, 4)$.