EXAMPLES OF SEMIALGEBRAIC CONVEX BODIES WITH NON-SEMIALGEBRAIC INTERSECTION BODY

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Abstract. We present an explicit construction of a semialgebraic convex body whose intersection body is not semialgebraic.

Let $K \subseteq \mathbb{R}^n$ be a convex body. The intersection body $\mathrm{I}K$ of K is the star-body defined by the radial function

$$\begin{aligned} \rho_{\mathrm{I}K} : \mathbb{R}^n &\to \mathbb{R} \\ u &\mapsto \frac{1}{\|u\|}\mathrm{vol}_{n-1}(u^\perp \cap K). \end{aligned}$$

Intersection bodies are studied in convex geometry and play a major role in the study of the Busemann-Petty problem [Lut88, Gar94, GKS99]. We refer to [Gar95] for the theory.

Convex algebraic geometry is interested in the study of geometric features of semialgebraic convex bodies. In particular, a question of interest concerns whether the intersection body of a semialgebraic convex body is semialgebraic. It is known that this is the case for convex bodies in $\mathbb{R}^2$ [Gar95, Thm. 8.1.4] and for polytopes in any dimension [BBMS22]. In this note, we provide examples showing that in $\mathbb{R}^n$ for $n \geq 3$ there are examples of convex bodies whose intersection body is not-semialgebraic.

Let $\Omega \subseteq \mathbb{R}^n$ be a connected open set. Let $F : \Omega \to \mathbb{R}$ be a function. We say that F is algebraic if $F(x_1, \dots, x_n)$ is solution of a non-trivial univariate polynomial equation with coefficients in $\mathbb{R}[x_1, \dots, x_n]$. Notice that if F is algebraic and $L \subseteq \mathbb{R}^n$ is a linear space, then $F|_{L\cap\Omega}$ is algebraic as well. If F is not algebraic, we say that it is transcendental.

We point out that composition of algebraic functions is algebraic. Moreover, if $F : \mathbb{R} \to \mathbb{R}$ is transcendental and $G : \mathbb{R}^n \to \mathbb{R}$ is algebraic and non-constant, then $F \circ G : \mathbb{R}^n \to \mathbb{R}$ is transcendental.

A consequence of the Projection Theorem for semialgebraic sets [BCR13, Sec. 2.2] is that a convex body is semialgebraic if and only if its support function, and equivalently its radial function are semialgebraic functions, in the sense that their graph is a semialgebraic set. In particular, if a convex body is semialgebraic, a consequence of the implicit function theorem is that the restriction of its radial function to any linear space is (locally on a dense open set) an algebraic function.

Given a function $f : [0, 1] \to \mathbb{R}$, define

$$K_f = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq f(z)^2, 0 \leq x \leq 1 - z\} \subseteq \mathbb{R}^3.$$

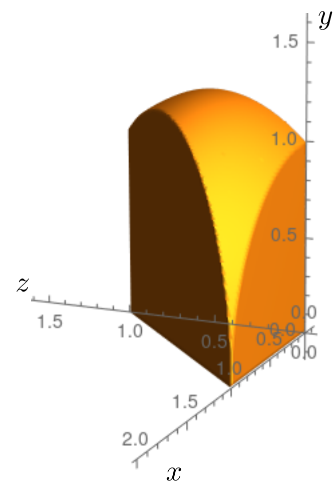

Figure 1. Example of convex body K_f. The intersection of K_f with the vertical plane $\{z = z_0\}$ is a section of a disc of radius $f(z_0)$ for $z_0 \in [0,1]$.

If f is concave and positive, then K_f is a convex body. Moreover, if f is algebraic, then K_f is semialgebraic. The main result of this note is the following:

Theorem 1. *If $f : [0,1] \to \mathbb{R}$ is an algebraic function such that the integral function $F(t) = \int_0^t f(s)ds$ is transcendental, then $\mathrm{I}K_f$ is not semialgebraic.*

Proof. We consider the restriction of the radial function of $\mathrm{I}K_f$ to the pencil of hyperplanes of the form

$$H_\alpha = \left\{ \eta \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \\ 0 \end{pmatrix} + \zeta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} : (\eta, \zeta) \in \mathbb{R}^2 \right\}.$$

Let $\varphi_\alpha : \mathbb{R}^2 \to \mathbb{R}^3$ be the isometric parametrization

$$\varphi_\alpha(\eta, \zeta) = \eta \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \\ 0 \end{pmatrix} + \zeta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

of H_α. It suffices to show that the function

$$\mathrm{vol}_2(H_\alpha \cap K_f) = \mathrm{vol}_2(\varphi_\alpha^{-1}(K_f))$$

is a transcendental function of $\cos(\alpha)$ (or equivalently $\sin(\alpha)$).

One can verify that

$$\varphi_\alpha^{-1}(K_f) = \left\{ (\eta, \zeta) : \eta \leq f(\zeta), 0 \leq \eta \leq \frac{1-\zeta}{\cos(\alpha)} \right\}.$$

In other words, $\varphi_\alpha^{-1}(K_f)$ is the convex region below the graph of the function

$$g_\alpha : [0,1] \to \mathbb{R}$$
$$\zeta \mapsto \begin{cases} f(\zeta) & \text{if } \zeta \leq \zeta_\alpha \\ \frac{1-\zeta}{\cos(\alpha)} & \text{if } \zeta \geq \zeta_\alpha, \end{cases}$$

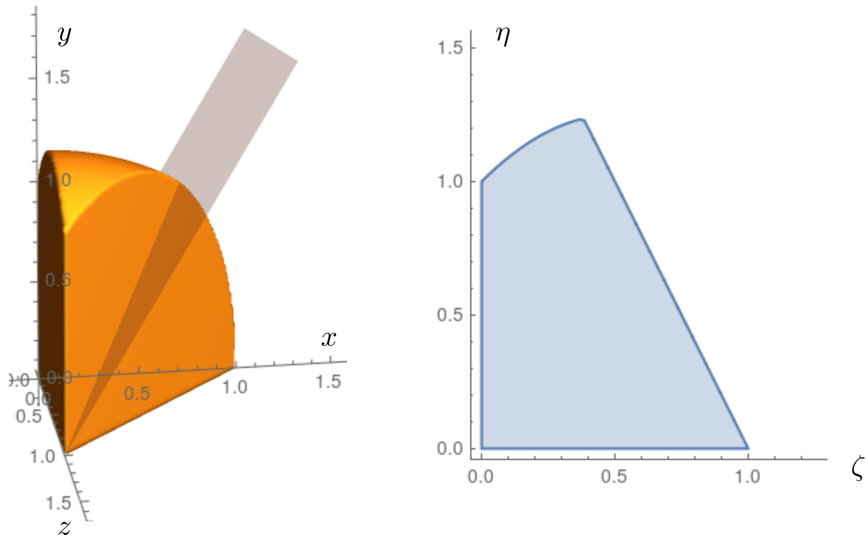

Figure 2. The linear cut with the hyperplane H_α and the resulting convex body $\varphi_\alpha^{-1}(K_f) \subseteq \mathbb{R}^2$. The peak is at $\zeta = \zeta_\alpha$.

where ζ_α is the unique value in $[0, 1]$ such that $f(\zeta_\alpha) = \frac{1-\zeta_\alpha}{\cos(\alpha)}$. In particular ζ_α is an algebraic function of $\cos(\alpha)$ because f is algebraic.

We obtain

$$\mathrm{vol}_2(H_\alpha \cap K_f) = \mathrm{vol}_2(\varphi_\alpha^{-1}(K_f)) = \int_0^1 g_\alpha(\eta)d\eta =$$
$$= \int_0^{\zeta_\alpha} f(\eta)d\eta + \int_{\zeta_\alpha}^1 \frac{1-\eta}{\cos(\alpha)}d\eta = F(\zeta_\alpha) + G(\zeta_\alpha).$$

Notice that $F(\zeta_\alpha)$ is a transcendental function of $\cos(\alpha)$, because F is transcendental and ζ_α is algebraic. On the other hand $G(\zeta_\alpha)$ is an algebraic function of $\cos(\alpha)$.

We deduce that $\mathrm{vol}_2(H_\alpha \cap K_f)$ is a transcendental function of $\cos(\alpha)$ and this concludes the proof. □

We record the following consequence of Theorem 1.

Corollary 2. *For every $n \geq 3$, there exists a semialgebraic convex body $K \geq \mathbb{R}^n$ such that the intersection body $\mathrm{I}K$ is not semialgebraic.*

Proof. Theorem 1 provides examples for $n = 3$. In general, it suffices to consider $K_f \times [0, 1]^{\times(n-3)} \subseteq \mathbb{R}^n$. □

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